

# On the Picard–Fuchs Equations for Massive $N = 2$ Seiberg–Witten Theories

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## ABSTRACT

A new method to obtain the Picard–Fuchs equations of effective,  $N = 2$  supersymmetric gauge theories with massive matter hypermultiplets in the fundamental representation is presented. It generalises a previously described method to derive the Picard–Fuchs equations of both pure super Yang–Mills and supersymmetric gauge theories with massless matter hypermultiplets. The techniques developed are well suited to symbolic computer calculations.

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## 1. Introduction.

New insights into the behaviour of low energy supersymmetric gauge theories have led to an explosion of activity in the subject [1]. One aspect which has attracted attention is the low-energy properties of the Coulomb phase of  $N=2$  gauge theories with  $N_f$  matter multiplets in the fundamental representation, with and without bare masses. Since the moduli space of these gauge theories coincides with that of a particular hyperelliptic curve [1]–[15], [23, 24], one attempts to extract the strong-coupling physics from the curve describing the theory in question. The desired information can be found from the prepotential  $\mathcal{F}(\vec{a})$  characterising the low-energy effective Lagrangian, and from the Seiberg–Witten (SW) period integrals

$$\vec{\Pi} = \begin{pmatrix} \vec{a}_D \\ \vec{a} \end{pmatrix}. \quad (1.1)$$

These are related to  $\mathcal{F}(\vec{a})$  by

$$a_D^i = \frac{\partial \mathcal{F}}{\partial a_i}. \quad (1.2)$$

In the weak-coupling region one can write (schematically)

$$\mathcal{F}(a) = \mathcal{F}_{\text{1-loop}}(a) + \mathcal{F}_{\text{instantons}}(a). \quad (1.3)$$

A number of different methods have emerged for evaluating equation (1.3) for a variety of groups, either using the properties of hyperelliptic curves, or from the microscopic Lagrangian itself [30]. However, the available results for the strong-coupling regions are still rather sparse, and as yet no complete picture has emerged, except for gauge groups of rank 1 or 2.

One strategy for obtaining strong-coupling information is to derive and solve a set of Picard–Fuchs (PF) equations for the SW period integrals. The PF equations have been derived in a number of special cases for  $N_f = 0$  and massless multiplets for  $N_f \neq 0$  [8, 19, 20, 21, 29].

The PF equations have also been formulated [20, 21, 29] as an explicit, analytic set of PF equations valid for arbitrary classical gauge groups, and an extensive set of values for  $N_f$  massless multiplets in the fundamental representation. A systematic method for finding PF equations for cases with bare mass  $m = 0$ , which is particularly convenient for symbolic computer computations, was given by Isidro *et al* [21]. Alternative techniques, not involving PF equations, are explored in [16, 17].

The situation for  $m \neq 0$  is less fully developed [1, 18, 22, 27, 28]. Part of the problem is that the preferred SW differential  $\lambda_{SW}$ , is an abelian differential of the third kind if  $m \neq 0$ , while of the second-kind if  $m = 0$ .

The SW differential has the property that

$$a_i = \oint_{\gamma_i} \lambda_{SW}, \quad a_i^D = \oint_{\gamma_i^D} \lambda_{SW}, \quad (1.4)$$

where  $\gamma_i$  and  $\gamma_i^D$  are closed 1-cycles with canonical intersection in a homology basis. Since  $\lambda_{SW}$  is of the third-kind when  $m \neq 0$ , *i.e.*, it has poles with non-vanishing residues, some care is required to formulate the PF equations. Explicit PF equations for  $SU(2)$  and  $SU(3)$  have been presented for  $m \neq 0$  [22, 28]. Even for these simplest examples several pages are required to write out the actual PF equations. Thus, it does not seem practical to present explicit PF equations for arbitrary gauge groups and  $N_f$  consistent with asymptotic freedom when  $m \neq 0$ . Rather, it seems more convenient to present a comprehensive set of explicit algorithms, from which one may obtain the PF equations with  $m \neq 0$ .

In this paper we generalise the results of [21], and present a new method to obtain the PF equations for arbitrary classical gauge groups, with massive matter hypermultiplets in the fundamental representation. We have obtained the explicit PF equations for a number of gauge groups with massive multiplets. The corresponding PF equations turn out to be extremely lengthy. Therefore, there is no

virtue in presenting these results. However, explicit results using our methods coincide with PF equations previously presented in the literature for  $m \neq 0$ . Thus our paper concentrates on a description of our method. In section 2 we formulate the problem, while in section 3 a necessary set of recursion relations are exhibited. Sections 4 and 5 describe how the PF equations are derived from this information. Final comments, as well as a summary and conclusions, appear in sections 6 and 7, respectively.

## 2. Formulation of the problem.

The strategy used in [21] to derive the PF equations of effective  $N = 2$  supersymmetric Yang–Mills theories in 4 dimensions can be modified to include massive matter hypermultiplets. To do so, let us first recall the necessary elements from [21].

Let us consider an effective  $N = 2$  supersymmetric Yang–Mills theory characterised by a certain gauge group  $G$  with rank  $r$ , a number  $N_f$  of massive matter hypermultiplets with bare masses  $m_j$ , where  $1 \leq j \leq N_f$ , and a number of moduli  $u_i$ , where  $1 \leq i \leq r$ . The matter hypermultiplets will be taken in the fundamental representation, and  $N_f$  will be restricted to those values for which the theory is asymptotically free, but for the moment  $G$  will remain unspecified. Consider the complex algebraic curves

$$y^2 = P(x; u_i; m_j; \Lambda), \tag{2.1}$$

where  $P$  is a polynomial in  $x$  of degree  $2g + 2$ , and  $\Lambda$  is the dynamically generated quantum scale. When all roots of  $P$  are pairwise different, *i.e.*, away from the zero locus of the discriminant of the curve, equation (2.1) defines a family of non-singular hyperelliptic Riemann surfaces  $\Sigma_g$  of genus  $g$  [31]. Under an appropriate choice of the polynomial  $P$ , the moduli space of quantum vacua of the theory under consideration is coincident with that of the curves defined by equation (2.1). On

$\Sigma_g$  there are  $g$  holomorphic 1-forms

$$x^j \frac{dx}{y}, \quad j = 0, 1, \dots, g-1. \quad (2.2)$$

The following  $g$  1-forms are meromorphic on  $\Sigma_g$  and have pole singularities at infinity of order greater than 1<sup>★</sup>

$$x^j \frac{dx}{y}, \quad j = g+1, g+2, \dots, 2g. \quad (2.3)$$

Furthermore, the 1-form

$$x^g \frac{dx}{y} \quad (2.4)$$

is meromorphic on  $\Sigma_g$ , with a simple pole at infinity. Altogether, the abelian differentials  $x^j dx/y$  in equations (2.2) and (2.3) will be denoted collectively by  $\omega_j$ , where  $j = 0, 1, \dots, 2g$ ,  $j \neq g$ . We define the *basic range*  $R$  to be  $R = \{0, 1, \dots, \check{g}, \dots, 2g\}$ , where a check over  $g$  means the value  $g$  is to be omitted.

Following [21] we give two definitions. Let us call  $W = y^2 = P(x; u_i; m_j; \Lambda)$  in equation (2.1). Moreover, given any differential  $x^n dx/y$ , with  $n \geq 0$  an integer, let us define its *generalised  $\mu$ -period*  $\Omega_n^{(\mu)}(u_i; m_j; \Lambda; \gamma)$  along a fixed 1-cycle  $\gamma \in H_1(\Sigma_g)$  as the line integral [32]

$$\Omega_n^{(\mu)}(u_i; m_j; \Lambda; \gamma) := (-1)^{\mu+1} \Gamma(\mu+1) \oint_{\gamma} \frac{x^n}{W^{\mu+1}} dx. \quad (2.5)$$

In equation (2.5),  $\Gamma$  stands for Euler's gamma function, while  $\gamma \in H_1(\Sigma_g)$  is any closed 1-cycle on the surface. For the sake of simplicity, we will drop  $u_i$ ,  $m_j$ ,  $\Lambda$  and  $\gamma$  from the notation for the periods  $\Omega_n^{(\mu)}$ . As explained in [21], we will work with an arbitrary value of  $\mu$ , which will only be set to  $-1/2$  at the very end.

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★ The differential  $x^{g+1} dx/y$  is of the second kind, *i.e.*, it has vanishing residues, for the particular curves of Seiberg-Witten theories. The differentials  $x^n dx/y$  have non-vanishing residues at infinity when  $n > g+1$ . This corrects a mistake in the terminology of [21].

In effective  $N = 2$  supersymmetric gauge theories, there exists a preferred differential, called the *Seiberg–Witten (SW) differential*,  $\lambda_{SW}$ , with the following property [1]: the electric and magnetic masses  $a_i$  and  $a_i^D$  entering the BPS mass formula are given by the periods of  $\lambda_{SW}$  along some specified closed cycles  $\gamma_i, \gamma_i^D \in H_1(\Sigma_g)$ , as in equation (1.4). In these theories, the polynomial  $P(x; u_i; m_j; \Lambda)$  is of the special form  $P = p^2(x) - G(x)$ , for certain  $p(x)$  and  $G(x)$  (see below). Then, the SW differential takes on the following expression:

$$\lambda_{SW} = \frac{x}{y} \left( \frac{G'}{G} \frac{p}{2} - p' \right) dx. \quad (2.6)$$

The SW differential further enjoys the property that its modular derivatives  $\partial \lambda_{SW} / \partial u_i$  are (linear combinations of the) holomorphic 1-forms [1]. This ensures positivity of the Kähler metric on moduli space.

### 3. The recursion relations.

As seen in the massless case treated in [21], the periods  $\Omega_n^{(\mu)}$  defined in equation (2.5) satisfy a set of recursion relations in both indices  $n$  and  $\mu$  that can be used to derive the PF equations. Similar conclusions continue to hold for the massive case as well. However, the polynomial  $P(x; u_i; m_j; \Lambda)$  defining the curve depends on the particular gauge group  $G$  in such a way that a general expression valid for all  $G$ , such as the one used in [21], cannot be given. Instead, the different gauge groups have to be treated separately. As the derivation of the recursion relations follows the same pattern used in [21] for the massless case, it will not be reproduced here. We will simply list the final results below.

$$a) \ G = SU(N_c), \ N_f < N_c.$$

The curve is given by [23]

$$W = p^2(x) - \Lambda_{N_f}^{2N_c - N_f} \prod_{j=1}^{N_f} (x + m_j), \quad (3.1)$$

where

$$p(x) = \sum_{i=0}^{N_c} u_i x^i, \quad u_{N_c} = 1, \quad u_{N_c-1} = 0. \quad (3.2)$$

Defining the symmetric polynomials in the masses  $S_{N_f-j}(m)$  through the expansion

$$\prod_{j=1}^{N_f} (x + m_j) = \sum_{j=0}^{N_f} S_{N_f-j}(m) x^j, \quad (3.3)$$

one finds that the following recursion relations hold:

$$\begin{aligned} \Omega_{n+2N_c}^{(\mu+1)} &= \Lambda_{N_f}^{2N_c-N_f} \sum_{j=0}^{N_f} S_{N_f-j} \Omega_{n+j}^{(\mu+1)} - (1+\mu) \Omega_n^{(\mu)} \\ &\quad - \sum_{i=0}^{N_c-1} \sum_{j=0}^{N_c-1} u_i u_j \Omega_{n+i+j}^{(\mu+1)} - 2 \sum_{j=0}^{N_c-1} u_j \Omega_{N_c+n+j}^{(\mu+1)} \end{aligned} \quad (3.4)$$

and

$$\begin{aligned} \Omega_n^{(\mu)} &= \frac{-1}{n+1-2N_c(1+\mu)} \left[ \Lambda_{N_f}^{2N_c-N_f} \sum_{j=0}^{N_f} (2N_c-j) S_{N_f-j} \Omega_{n+j}^{(\mu+1)} \right. \\ &\quad \left. + \sum_{j=0}^{N_c-1} \sum_{l=0}^{N_c-1} (j+l-2N_c) u_j u_l \Omega_{n+j+l}^{(\mu+1)} + 2 \sum_{j=0}^{N_c-1} (j-N_c) u_j \Omega_{N_c+n+j}^{(\mu+1)} \right]. \end{aligned} \quad (3.5)$$

When  $n+1-2N_c(2+\mu) \neq 0$ , one can combine equations (3.4) and (3.5) to obtain, after shifting  $n+2N_c \rightarrow n$ ,

$$\begin{aligned} \Omega_n^{(\mu+1)} &= \frac{1}{n+1-2N_c(2+\mu)} \times \\ &\quad \left[ \Lambda_{N_f}^{2N_c-N_f} \sum_{j=0}^{N_f} (n-2N_c+1-j(1+\mu)) S_{N_f-j} \Omega_{n-2N_c+j}^{(\mu+1)} \right. \\ &\quad + 2 \sum_{j=0}^{N_c-1} ((1+\mu)(N_c+j) - (n-2N_c+1)) u_j \Omega_{n-N_c+j}^{(\mu+1)} \\ &\quad \left. + \sum_{j=0}^{N_c-1} \sum_{l=0}^{N_c-1} ((j+l)(1+\mu) - (n-2N_c+1)) u_j u_l \Omega_{n+j+l-2N_c}^{(\mu+1)} \right]. \end{aligned} \quad (3.6)$$

Modular derivatives of periods are given by

$$\frac{\partial \Omega_n^{(\mu)}}{\partial u_i} = 2 \sum_{j=0}^{N_c} u_j \Omega_{n+i+j}^{(\mu+1)}. \quad (3.7)$$

*b* )  $G = SU(N_c)$ ,  $2 < N_c \leq N_f < 2N_c$ .

The curve is given by [23]

$$W = \left[ p(x) + \frac{1}{4} \Lambda_{N_f}^{2N_c-N_f} \sum_{j=0}^{N_f-N_c} S_j x^{N_f-N_c-j} \right]^2 - \Lambda_{N_f}^{2N_c-N_f} \prod_{j=1}^{N_f} (x + m_j), \quad (3.8)$$

with  $p(x)$  as in equation (3.2). We observe that this curve can be obtained from the one given in equations (3.1) and (3.2) by a shift of some of the moduli  $u_i$ :

$$u_i \rightarrow u_i + \frac{1}{4} \Lambda_{N_f}^{2N_c-N_f} S_{N_f-N_c-i}, \quad 0 \leq i \leq N_f - N_c. \quad (3.9)$$

In the classical limit, this redefinition does not affect the moduli. Applying this shift to equations (3.4) through (3.7) we can straightforwardly derive the recursion relations corresponding to this case from those of the previous case. Alternatively, one could carry out a step-by-step derivation like the one needed for  $N_f < N_c$ . Either way, the recursion relations turn out to be given by

$$\begin{aligned} \Omega_{n+2N_c}^{(\mu+1)} &= \Lambda_{N_f}^{2N_c-N_f} \sum_{j=0}^{N_f} S_{N_f-j} \Omega_{n+j}^{(\mu+1)} - \sum_{j=0}^{N_c-1} \sum_{l=0}^{N_c-1} u_j u_l \Omega_{n+j+l}^{(\mu+1)} \\ &- 2 \sum_{j=0}^{N_c-1} u_j \Omega_{N_c+n+j}^{(\mu+1)} - \frac{1}{2} \Lambda_{N_f}^{2N_c-N_f} \sum_{l=0}^{N_c} \sum_{i=0}^{N_f-N_c} u_l S_i \Omega_{N_f-N_c+n-i+l}^{(\mu+1)} \\ &- \frac{1}{16} \Lambda_{N_f}^{4N_c-2N_f} \sum_{j=0}^{N_f-N_c} \sum_{l=0}^{N_f-N_c} S_j S_l \Omega_{2N_f-2N_c+n-j-l}^{(\mu+1)} - (1+\mu) \Omega_n^{(\mu)} \end{aligned} \quad (3.10)$$



and

$$\begin{aligned}
\Omega_n^{(\mu)} &= \frac{-1}{n+1-2N_c(1+\mu)} \left[ \Lambda_{N_f}^{2N_c-N_f} \sum_{j=0}^{N_f} (2N_c-j) S_{N_f-j} \Omega_{n+j}^{(\mu+1)} \right. \\
&+ 2 \sum_{j=0}^{N_c-1} (j-N_c) u_j \Omega_{N_c+n+j}^{(\mu+1)} + \sum_{j=0}^{N_c-1} \sum_{l=0}^{N_c-1} (j+l-2N_c) u_j u_l \Omega_{n+j+l}^{(\mu+1)} \\
&+ \frac{1}{16} \Lambda_{N_f}^{4N_c-2N_f} \sum_{j=0}^{N_f-N_c} \sum_{l=0}^{N_f-N_c} (2N_f-4N_c-j-l) S_j S_l \Omega_{n+2N_f-2N_c-j-l}^{(\mu+1)} \\
&\left. + \frac{1}{2} \Lambda_{N_f}^{2N_c-N_f} \sum_{l=0}^{N_c} \sum_{j=0}^{N_f-N_c} (N_f-3N_c-j+l) u_l S_j \Omega_{n+N_f-N_c-j+l}^{(\mu+1)} \right]. \tag{3.11}
\end{aligned}$$

When  $n+1-2N_c(2+\mu) \neq 0$ , one can combine equations (3.10) and (3.11) to obtain, after shifting  $n+2N_c \rightarrow n$ ,

$$\begin{aligned}
\Omega_n^{(\mu+1)} &= \frac{1}{n+1-2N_c(2+\mu)} \left[ \Lambda_{N_f}^{2N_c-N_f} \sum_{j=0}^{N_f} (n-2N_c+1-j(1+\mu)) S_{N_f-j} \Omega_{n-2N_c+j}^{(\mu+1)} \right. \\
&- \frac{\Lambda_{N_f}^{4N_c-2N_f}}{16} \sum_{l=0}^{N_f-N_c} \sum_{j=0}^{N_f-N_c} (n-2N_c+1-(1+\mu)(2N_f-2N_c-j-l)) S_j S_l \Omega_{n+2N_f-4N_c-j-l}^{(\mu+1)} \\
&- \frac{1}{2} \Lambda_{N_f}^{2N_c-N_f} \sum_{l=0}^{N_c} \sum_{j=0}^{N_f-N_c} (n-2N_c+1-(1+\mu)(l-j+N_f-N_c)) u_l S_j \Omega_{n+N_f-3N_c-j+l}^{(\mu+1)} \\
&- \sum_{j=0}^{N_c-1} \sum_{l=0}^{N_c-1} (n-2N_c+1-(1+\mu)(j+l)) u_j u_l \Omega_{n-2N_c+j+l}^{(\mu+1)} \\
&\left. - 2 \sum_{j=0}^{N_c-1} (n-2N_c+1-(1+\mu)(j+N_c)) u_j \Omega_{n-N_c+j}^{(\mu+1)} \right]. \tag{3.12}
\end{aligned}$$

Modular derivatives of periods are given by

$$\frac{\partial \Omega_n^{(\mu)}}{\partial u_i} = 2 \sum_{j=0}^{N_c} u_j \Omega_{n+i+j}^{(\mu+1)} + \frac{1}{2} \Lambda_{N_f}^{2N_c-N_f} \sum_{j=0}^{N_f-N_c} S_j \Omega_{N_f-N_c+n-j+i}^{(\mu+1)}. \tag{3.13}$$

$c$  )  $SO(N_c)$ ,  $N_f < N_c - 2$ .

The curve for  $N_f < N_c - r - 2$  is given by [24]

$$W = p^2(x) - \Lambda_{N_f}^{2(N_c - N_f - 2)} x^d \prod_{j=1}^{N_f} (x^2 - m_j^2), \quad (3.14)$$

with the rank  $r$  and the power  $d$  being respectively given by  $r = N_c/2$  and  $d = 4$ , if  $N_c$  is even, and  $r = (N_c - 1)/2$  and  $d = 2$ , if  $N_c$  is odd. The polynomial  $p(x)$  is given by

$$p(x) = \sum_{j=0}^{2r} u_j x^j, \quad u_{2r} = 1, \quad u_{\text{odd}} = 0. \quad (3.15)$$

Let us expand the mass term of equation (3.14) in terms of the symmetric polynomials in the *squared* masses  $T_{N_f-j}(m^2)$  as follows:

$$\prod_{j=1}^{N_f} (x^2 - m_j^2) = \sum_{j=0}^{N_f} (-1)^{N_f-j} T_{N_f-j}(m^2) x^{2j}. \quad (3.16)$$

The above expansion could just as well be expressed as a double summation with coefficients given by the symmetric polynomials in the masses  $S_{N_f-j}(m)$ , as done for  $SU(N_c)$  in equation (3.3). However, we will find it more convenient to use the expansion (3.16), as it manifestly preserves the even parity of the  $SO(N_c)$  curve under  $x \rightarrow -x$ . Reasoning as in [21], one finds that the following recursion relations hold:

$$\begin{aligned} \Omega_n^{(\mu)} &= \frac{-1}{n+1-4r(1+\mu)} \left[ \Lambda_{N_f}^{2(N_c-N_f-2)} \sum_{i=0}^{N_f} (-1)^{N_f-i} (4r-2i-d) T_{N_f-i} \Omega_{n+d+2i}^{(\mu+1)} \right. \\ &\quad \left. + \sum_{i=0}^{2r-2} \sum_{j=0}^{2r-2} (i+j-4r) u_i u_j \Omega_{n+i+j}^{(\mu+1)} + 2 \sum_{j=0}^{2r-2} (j-2r) u_j \Omega_{n+j+2r}^{(\mu+1)} \right] \end{aligned} \quad (3.17)$$

and

$$\begin{aligned}
\Omega_n^{(\mu+1)} &= \frac{1}{n+1-4r(2+\mu)} \times \\
&\left[ \Lambda_{N_f}^{2(N_c-N_f-2)} \sum_{i=0}^{N_f} (-1)^{N_f-i} (n-4r+1-(1+\mu)(2i+d)) T_{N_f-i} \Omega_{n+d-4r+2i}^{(\mu+1)} \right. \\
&+ 2 \sum_{j=0}^{2r-2} ((1+\mu)(j+2r)-(n-4r+1)) u_j \Omega_{n-2r+j}^{(\mu+1)} \\
&\left. + \sum_{i=0}^{2r-2} \sum_{j=0}^{2r-2} ((1+\mu)(i+j)-(n-4r+1)) u_i u_j \Omega_{n+i+j-4r}^{(\mu+1)} \right].
\end{aligned} \tag{3.18}$$

Modular derivatives of periods are given by

$$\frac{\partial \Omega_n^{(\mu)}}{\partial u_i} = 2 \sum_{j=0}^{2r} u_j \Omega_{n+i+j}^{(\mu+1)}. \tag{3.19}$$

It is known from [24] that the curve for  $N_c - r - 2 \leq N_f < N_c - 2$  can be obtained from that for  $N_f < N_c - r - 2$  by a shift of the moduli similar to the one performed in equation (3.9). The necessary shift now affects the even moduli only, and it involves the symmetric polynomials in the squared masses  $T_j(m^2)$  rather than the  $S_j(m)$ . Application of this shift to the recursion relations given in equations (3.17), (3.18) and (3.19) will produce the recursions corresponding to  $N_c - r - 2 \leq N_f < N_c - 2$ .

$d$  )  $Sp(N_c)$ ,  $0 \leq N_f < N_c + 2$ .

The curve is given by [5, 25]<sup>★</sup>

$$x^2 W = p^2(x) - \Lambda_{N_f}^{2(2r+2-N_f)} \prod_{j=1}^{N_f} (x^2 + m_j^2), \tag{3.20}$$

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★ The sign of the  $m^2$  term in ref. [25] is opposite to that of eqn. (3.20). If so, that would mean the double scaling limit would only apply to an even  $N_f$ . The replacement  $T_{N_f-j}(m^2) \rightarrow (-1)^{N_f-j} T_{N_f-j}(m^2)$  in eqns. (3.22), (3.23) and (3.24) below will accommodate the sign given in ref. [25].

where  $N_c = 2r$ . The polynomial  $p(x)$  is given by

$$p(x) = \sum_{j=2}^{2r+2} u_j x^j + \Lambda_{N_f}^{2r+2-N_f} \prod_{j=1}^{N_f} m_j, \quad u_{2r+2} = 1, \quad u_{\text{odd}} = 0. \quad (3.21)$$

The mass term of equation (3.20) can be expanded in terms of the symmetric polynomials  $T_{N_f-j}(m^2)$  defined by

$$\prod_{j=1}^{N_f} (x^2 + m_j^2) = \sum_{j=0}^{N_f} T_{N_f-j}(m^2) x^{2j}. \quad (3.22)$$

Similar steps to the ones taken above then lead to the following recursion relations:

$$\begin{aligned} \Omega_n^{(\mu)} &= \frac{-1}{n+1-(4r+2)(1+\mu)} \left[ \Lambda_{N_f}^{2(2r+2-N_f)} \sum_{j=1}^{N_f} (4(r+1)-2j) T_{N_f-j} \Omega_{n+2j-2}^{(\mu+1)} \right. \\ &+ 2 \sum_{i=2}^{2r} (i-2(r+1)) u_i \Omega_{n+2r+i}^{(\mu+1)} + \sum_{i=2}^{2r} \sum_{j=2}^{2r} (i+j-4(r+1)) u_i u_j \Omega_{n+i+j-2}^{(\mu+1)} \\ &\left. + 2 \Lambda_{N_f}^{2r+2-N_f} \prod_{i=1}^{N_f} m_j \sum_{j=2}^{2r+2} (j-4(r+1)) u_j \Omega_{n+j-2}^{(\mu+1)} \right] \end{aligned} \quad (3.23)$$

and

$$\begin{aligned} \Omega_n^{(\mu+1)} &= \frac{1}{n+1-(4r+2)(2+\mu)} \times \\ &\left[ \sum_{i=2}^{2r} \sum_{j=2}^{2r} ((i+j-2)(1+\mu) - (n-4r-1)) u_i u_j \Omega_{n+i+j-4r-4}^{(\mu+1)} \right. \\ &+ 2 \sum_{i=2}^{2r} ((1+\mu)(2r+i) - (n-4r-1)) u_i \Omega_{n-2r+i-2}^{(\mu+1)} \\ &+ 2 \Lambda_{N_f}^{2r+2-N_f} \prod_{i=1}^{N_f} m_j \sum_{j=2}^{2r+2} ((1+\mu)(j-2) - (n-4r-1)) u_j \Omega_{n+j-4r-4}^{(\mu+1)} \\ &\left. + \Lambda_{N_f}^{2(2r+2-N_f)} \sum_{j=1}^{N_f} ((n-4r-1) + (1+\mu)(2-2j)) T_{N_f-j} \Omega_{n+2j-4r-4}^{(\mu+1)} \right]. \end{aligned} \quad (3.24)$$

Modular derivatives of periods are given by

$$\frac{\partial \Omega_n^{(\mu)}}{\partial u_i} = 2 \sum_{j=2}^{2r+2} u_j \Omega_{n+i+j-2}^{(\mu+1)} + 2\Lambda_{N_f}^{2r+2-N_f} \prod_{j=1}^{N_f} m_j \Omega_{n+i-2}^{(\mu+1)}. \quad (3.25)$$

#### 4. Derivation of the Picard–Fuchs equations.

Following the steps of [21], one can use the recursion relations of the previous section to derive a coupled system of first-order, partial differential equations (with respect to the moduli) satisfied by the periods. We first set  $\mu = -1/2$  in all what follows. Then we need to identify the appropriate subspace of periods that one must restrict to, in order to properly solve the above recursions. We recall that a key element is the behaviour of the curves under the operation of parity  $x \rightarrow -x$ . A glance at the equations of the previous section immediately reveals that conclusions completely analogous to those of [21] continue to hold for the massive case. Let us briefly recall them.

- For the  $SO(N_c)$  gauge groups, one must restrict to the *even* subspace of the basic range  $R$ , *i.e.*, to even values of the subindex  $n$ . This follows from two facts. One is that all the odd Casimirs of the gauge group  $SO(N_c)$  vanish, so the recursion relations have a step of 2 units. Furthermore, the solution to those recursions can be expressed in terms of a set of initial data with an even value for the subindex. As a function of the rank  $r$ , the genus  $g$  of the  $SO(N_c)$  curve is  $g = 2r - 1$ . This being odd, the value of the subindex  $n$  at which the recursion (3.17) blows up is skipped. Similarly, the zero of the denominator of equation (3.18) is avoided when  $n$  is even.

- For the  $Sp(N_c)$  gauge groups, one must restrict to the *odd* subspace of the basic range  $R$ , *i.e.*, to odd values of the subindex  $n$ . Again, this follows from the same facts as above. All the odd Casimirs of the gauge group  $Sp(N_c)$  vanish, so the recursion relations have a step of 2 units, but the solution to those recursions can

be expressed in terms of a set of initial data with an odd value for the subindex. As a function of the rank  $r$ , the genus  $g$  of the  $Sp(N_c)$  curve is  $g = 2r$ . This being even, the value of the subindex  $n$  at which the recursion (3.23) blows up is skipped. Similarly, the zero of the denominator of equation (3.24) is avoided when  $n$  is odd. The factor of  $x^2$  present in the left-hand side of equation (3.20) is responsible for this odd parity, as opposed to the even parity of the  $SO(N_c)$  recursions.

- For the  $SU(N_c)$  gauge groups, the curve has no well defined parity under  $x \rightarrow -x$ . This is a consequence of the fact that  $SU(N_c)$  has both even and odd Casimirs which, in turn, causes the recursion relations have a step of 1 unit. To solve the recursions, one must first work on the enlarged subspace of periods given by  $R \cup \{g\}$ , *i.e.*,  $\Omega_n^{(\pm 1/2)}$ , with  $n \in R \cup \{g\}$ . This is in order to avoid the divergence of the recursion relations (3.5) and (3.11) that occurs when the subindex  $n$  takes on the value  $n = g$ , where the genus  $g$  now equals the rank  $r$ ,  $g = r$ . Next, one applies a linear relation satisfied by the  $\Omega_n^{(+1/2)}$ , where  $n \in R \cup \{g\}$ . For a derivation of this linear relation that also holds in the massive case treated here, see [21].

Once the correct subspace of periods has been identified, the recursion relations can be solved as explained in [21]. Let us arrange the periods  $\Omega_n^{(\pm 1/2)}$ , where  $n$  spans the appropriate subspace of  $R$ , as column vectors:  $\Omega^{(\pm 1/2)} = (\Omega_1^{(\pm 1/2)}, \dots, \Omega_r^{(\pm 1/2)}, \Omega_{r+1}^{(\pm 1/2)}, \dots, \Omega_{2r}^{(\pm 1/2)})^t$ . We have called the dimension of the subspace under consideration  $2r$ , as it turns out to equal two times the rank  $r$  of the gauge group. We have also arranged the entries of  $\Omega^{(\pm 1/2)}$  in such a way that the first  $r$  of them are holomorphic, while the last  $r$  of them are meromorphic. From the recursion relations, the vectors of periods  $\Omega^{(-1/2)}$  and  $\Omega^{(+1/2)}$  are linearly related through a matrix  $M$

$$\Omega^{(-1/2)} = M \cdot \Omega^{(+1/2)}, \quad (4.1)$$

where  $M$  is  $(2r \times 2r)$ -dimensional. Its entries are certain polynomial functions in the moduli  $u_i$ , the bare masses  $m_j$  and the quantum scale  $\Lambda$ , explicitly computable using the recursion relations given in the previous section.

Similarly, one can use the expressions for the modular derivatives (equations (3.7), (3.13), (3.19) and (3.25)), together with the recursions, in order to write a system of equations which, in matrix form, reads

$$\frac{\partial}{\partial u_i} \Omega^{(-1/2)} = D(u_i) \cdot \Omega^{(+1/2)}. \quad (4.2)$$

The matrix  $D(u_i)$  is  $(2r \times 2r)$ -dimensional. Again, its entries are some polynomial functions in the moduli  $u_i$ , the bare masses  $m_j$  and the quantum scale  $\Lambda$ , explicitly computable from the recursion relations. Assume for the moment that the matrix  $M$  of equation (4.1) is invertible. Combining the latter with equation (4.2) one ends up with

$$\frac{\partial}{\partial u_i} \Omega^{(-1/2)} = U_i \cdot \Omega^{(-1/2)}, \quad (4.3)$$

where we have defined the matrix  $U_i$  as

$$U_i = D(u_i) \cdot M^{-1}. \quad (4.4)$$

Equation (4.3) is a coupled system of first-order, partial differential equations satisfied by the periods  $\Omega^{(-1/2)}$ : *the first-order PF equations*. It expresses the modular derivatives of the basic periods  $\Omega^{(-1/2)}$  (of the subspace under consideration) as certain linear combinations of the same periods  $\Omega^{(-1/2)}$ . The coefficients entering those combinations (*i.e.*, the entries of the  $U_i$  matrices) are some rational functions of the moduli  $u_i$ , the masses  $m_j$  and the quantum scale  $\Lambda$ , explicitly computable from the recursion relations.

To close this section, let us comment on the invertibility of the matrix  $M$  of equation (4.1). Explicit evaluation in a wide class of examples shows  $\det M$  to be a product of the factors of the discriminant  $\Delta(u_i, m_j, \Lambda)$  of the corresponding curve, possibly with different multiplicities, and up to an overall non-zero constant. Therefore, the  $M$  matrix encodes the singularity structure of the curve, and it is invertible except at the singularities of moduli space, *i.e.*, except on the zero locus of

the discriminant  $\Delta(u_i, m_j, \Lambda)$  of the curve. This conclusion holds with some *caveat* when the gauge group is  $SU(N_c)$ , since  $\det M$  may then pick up some additional zeroes. This possible new zero locus of  $\det M$  occurs for the same reasons already explained in [21] for the massless case. The inclusion of non-zero bare masses  $m_j$  does not alter the arguments given in [21] as far as  $\det M$  is concerned, and they continue to hold in the massive case as well.

## 5. Decoupling the Picard–Fuchs equations.

In principle, integration of the system (4.3) yields the periods as functions of the moduli  $u_i$ . The particular 1-cycle  $\gamma \in H_1(\Sigma_g)$  being integrated over appears in the specific choice of boundary conditions that one makes. In practice, however, the fact that the system (4.3) is coupled makes it very difficult to solve. A possible strategy is to concentrate on one particular subset of periods and try to obtain a reduced system of equations satisfied by it, at the cost of increasing their order. In [21] we have made use of the fact that one can perform a change of basis that included  $\Omega_{SW}$ , the period of  $\lambda_{SW}$ , as a basic vector. The decoupling of the resulting equations then followed from the property that the modular derivatives of the SW differential  $\lambda_{SW}$  are the holomorphic differentials of the appropriate subspace within which the recursions are being solved—we call this the *potential property* of  $\lambda_{SW}$ . However, a similar change of basis is inconvenient now, because  $\lambda_{SW}$  is of the third kind. Let us see how this difficulty can be circumvented.

Consider the  $U_i$  matrix in equation (4.3) and block-decompose it as

$$U_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}, \quad (5.1)$$

where all four blocks  $A_i$ ,  $B_i$ ,  $C_i$  and  $D_i$  are  $r \times r$ . Next take the equations for the derivatives of the holomorphic periods,  $\partial\Omega_n/\partial u_i$ ,  $1 \leq n \leq r$ , and solve them for the meromorphic periods  $\Omega_n$ ,  $r \leq n \leq 2r$ , in terms of the holomorphic ones and



their modular derivatives. That is, consider<sup>★</sup>

$$\frac{\partial}{\partial u_i} \begin{pmatrix} \Omega_1 \\ \vdots \\ \Omega_r \end{pmatrix} - A_i \begin{pmatrix} \Omega_1 \\ \vdots \\ \Omega_r \end{pmatrix} = B_i \begin{pmatrix} \Omega_{r+1} \\ \vdots \\ \Omega_{2r} \end{pmatrix}. \quad (5.2)$$

Solving equation (5.2) for the meromorphic periods involves inverting the matrix  $B_i$ . Although we lack a formal proof that  $B_i$  is invertible, when the rank  $r$  of the gauge group is greater than 1,  $\det B_i$  turns out to have two types of zeroes on moduli space. The first zero locus contains a product of the factors of the discriminant  $\Delta(u_i, m_j, \Lambda)$ , possibly with different multiplicities. The second zero locus is unrelated to the discriminant. Away from these singularities,  $B_i$  is invertible so, from equation (5.2),

$$\begin{pmatrix} \Omega_{r+1} \\ \vdots \\ \Omega_{2r} \end{pmatrix} = B_i^{-1} \cdot \left( \frac{\partial}{\partial u_i} - A_i \right) \begin{pmatrix} \Omega_1 \\ \vdots \\ \Omega_r \end{pmatrix}. \quad (5.3)$$

Next, substitute the meromorphic periods (5.3) into the last  $r$  equations of (4.3),

$$\frac{\partial}{\partial u_i} \begin{pmatrix} \Omega_{r+1} \\ \vdots \\ \Omega_{2r} \end{pmatrix} = C_i \begin{pmatrix} \Omega_1 \\ \vdots \\ \Omega_r \end{pmatrix} + D_i \begin{pmatrix} \Omega_{r+1} \\ \vdots \\ \Omega_{2r} \end{pmatrix}, \quad (5.4)$$

and rearrange terms in order to obtain<sup>†</sup>

$$\begin{aligned} & \frac{\partial^2}{\partial u_i^2} \begin{pmatrix} \Omega_1 \\ \vdots \\ \Omega_r \end{pmatrix} - \left[ \frac{\partial B_i}{\partial u_i} B_i^{-1} + A_i + B_i D_i B_i^{-1} \right] \frac{\partial}{\partial u_i} \begin{pmatrix} \Omega_1 \\ \vdots \\ \Omega_r \end{pmatrix} \\ & + \left[ B_i D_i B_i^{-1} A_i - B_i C_i + \frac{\partial B_i}{\partial u_i} B_i^{-1} A_i - \frac{\partial A_i}{\partial u_i} \right] \begin{pmatrix} \Omega_1 \\ \vdots \\ \Omega_r \end{pmatrix} = 0. \end{aligned} \quad (5.5)$$

Equation (5.5) is a second-order coupled system, satisfied by the holomorphic pe-

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★ For notational simplicity we have dropped the superscript  $\mu = -1/2$ , with the understanding that it has been fixed.

† No summation over  $i$  is implied here.

riods on the curve: *the second-order PF equations*. In order to decouple them, one now employs the potential property of the SW differential  $\lambda_{SW}$ ,

$$\frac{\partial}{\partial u_i} \lambda_{SW} = \omega_i, \quad i = 1, \dots, r, \quad (5.6)$$

and from here one concludes an analogous property for the corresponding periods,

$$\frac{\partial}{\partial u_i} \Omega_{SW} = \Omega_i, \quad i = 1, \dots, r. \quad (5.7)$$

The passage from equation (5.6) to equation (5.7) is justified, even though in the presence of non-zero bare masses  $m_j$  the SW differential  $\lambda_{SW}$  is of the third kind. The reason is that, from [1], the residues of  $\lambda_{SW}$  are known to be some multiples of the bare masses  $m_j$ , and are therefore independent of the moduli  $u_i$ .

Finally, substitution of equation (5.7) into (5.5) produces a *decoupled* system of third-order, partial differential equations for the SW period  $\Omega_{SW}$ ,

$$\begin{aligned} \frac{\partial^2}{\partial u_i^2} \begin{pmatrix} \partial_1 \Omega_{SW} \\ \vdots \\ \partial_r \Omega_{SW} \end{pmatrix} - \left[ \frac{\partial B_i}{\partial u_i} B_i^{-1} + A_i + B_i D_i B_i^{-1} \right] \frac{\partial}{\partial u_i} \begin{pmatrix} \partial_1 \Omega_{SW} \\ \vdots \\ \partial_r \Omega_{SW} \end{pmatrix} \\ + \left[ B_i D_i B_i^{-1} A_i - B_i C_i + \frac{\partial B_i}{\partial u_i} B_i^{-1} A_i - \frac{\partial A_i}{\partial u_i} \right] \begin{pmatrix} \partial_1 \Omega_{SW} \\ \vdots \\ \partial_r \Omega_{SW} \end{pmatrix} = 0, \end{aligned} \quad (5.8)$$

which are *the third-order PF equations* of the massive  $N = 2$  theory.

## 6. Final Comments.

We observe that the recursion relations on which our method is founded can be derived without taking recourse to the SW differential  $\lambda_{SW}$  or its period  $\Omega_{SW}$ : all that is required is a knowledge of the curve. One first derives a set of recursion relations. Next one solves them, to obtain a first-order system (4.3) whose submatrices  $A_i$ ,  $B_i$ ,  $C_i$  and  $D_i$  contain all the relevant information. The final step is the decoupling procedure. Therefore, whatever limits we may want to take in the equations above must be taken at the level of the recursion relations. If the latter enjoy the correct limiting properties, so will the PF equations derived from them.

Two limits are worth taking. One is that in which all bare masses  $m_j$  tend to zero:  $m_j \rightarrow 0$ . This is the reduction to the massless case. The other one is the integrating out of one massive quark, also called *double-scaling limit*: sending the  $j$ -th mass to infinity,  $m_j \rightarrow \infty$ , and the quantum scale to zero,  $\Lambda_{N_f}^p \rightarrow 0$ , while keeping  $m_j \Lambda_{N_f}^p$  constant and setting it equal to the new quantum scale  $\Lambda_{N_f-1}^{p'}$ , ( $p$  and  $p'$  being the required powers to which  $\Lambda_{N_f}$  and  $\Lambda_{N_f-1}$  are raised, in the presence of  $N_f$  and  $N_f - 1$  flavours, respectively). This removes one flavour from the problem. One can easily check that all the recursion relations given in section 3 tend to the corresponding recursions (either massless, or with one massive quark less) in the appropriate limits. This is a trivial consequence of the fact that the massive curves themselves are so constructed as to reproduce both limits correctly. For the massless limit in particular, the recursion relations already derived in [21] are also correctly reproduced.

However, the limit in which all quarks become massless is more intriguing, in the following sense: the final equations for  $\Omega_{SW}$  are third-order in the massive case, while they are only second-order in the massless case. How does this reduction in the order of the equations take place? This point has been argued in [22], but let us see how it can be recast in our language.

As already remarked, the system of first-order equations (4.3) not only enjoys the correct limiting properties, but its derivation follows the same pattern in both

the massive and the massless case, since no use is made of the SW differential  $\lambda_{SW}$  or its period  $\Omega_{SW}$ . However, the passage from the first-order equations (4.3) to the second-order equations (5.5) (or equations (3.10) of [21]) differs in the massive and the massless cases. In both cases one solves for the meromorphic periods in terms of the holomorphic periods and their modular derivatives. However, in the massless case one can immediately apply the potential property of the SW period  $\Omega_{SW}$ , while in the massive case one still substitutes all the meromorphic periods into the remaining first-order equations. This further step must be taken because, the SW differential now being of the third kind, one cannot apply the potential property directly. This accounts for the increase in the order of the final equations.

The limit  $m_j \rightarrow 0$  in which all masses vanish must therefore be taken in the recursion relations, or at most at the level of the first-order equations (4.3), *i.e.*, prior to the decoupling of the equations. The reason is that the decoupling procedure does not commute with the limit, as it proceeds differently in the two cases.

## 7. Summary and Conclusions.

In this paper we have extended a previously described derivation of the Picard–Fuchs equations [21], in order to include the case of massive matter hypermultiplets in the fundamental representation. It is systematic, well suited for symbolic computer computations, and holds for any classical gauge group. Our method is based on a set of recursion relations satisfied by the period integrals that one can define on a hyperelliptic Riemann surface. We explicitly focused on the case of effective  $N = 2$  supersymmetric Yang–Mills theories in 4 dimensions, where the relevant Riemann surfaces are such that their moduli space coincides with the moduli space of quantum vacua of the theory under consideration. From this point of view, the Picard–Fuchs equations have proved to be an important tool in probing the structure of moduli space and in computing the full prepotential, including instantons.

For gauge groups with rank 2 or greater, the Picard–Fuchs equations in the presence of massive matter become intractable. The goal of explicitly computing the full quantum prepotential of effective  $N = 2$  theories may well have to be accomplished using alternative techniques, such as the ones put forward in [16, 17]. However, we believe the method presented here may find application elsewhere, as our derivation is not limited to these specific areas, and its algebraic nature lends itself easily to different uses.

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## APPENDIX

### Computer Calculation of the PF Equations

Let us return to equation (5.5), where we have observed that the entries of the blocks  $A_i$ ,  $B_i$ ,  $C_i$  and  $D_i$  are certain rational functions of the moduli  $u_i$ , the bare masses  $m_j$ , and the quantum scale  $\Lambda$ . This rational character stems from the inversion of the  $M$  matrix in equation (4.4), where division by  $\det M$  is required. As explained,  $\det M$  is also a polynomial in the moduli and the bare masses. Transferring  $\det M$  to the left-hand side of equation (4.3) leaves us with

$$\det M \frac{\partial}{\partial u_i} \Omega^{(-1/2)} = \tilde{U}_i \cdot \Omega^{(-1/2)}, \quad (\text{A.1})$$

where  $\tilde{U}_i$  is the matrix product  $D(u_i) \cdot \tilde{M}^{-1}$ ,  $\tilde{M}$  being the matrix of cofactors of  $M$ . Given that both  $D(u_i)$  and  $\tilde{M}$  have polynomial entries in the moduli and the masses,  $\tilde{U}_i$  also has purely polynomial entries, rather than rational functions.<sup>★</sup> To the effect of performing symbolic computer calculations, this has an obvious computational advantage, at the small expense of having to modify equations (4.3) through (5.5). We first block-decompose  $\tilde{U}_i$  in equation (A.1) as

$$\tilde{U}_i = \begin{pmatrix} \tilde{A}_i & \tilde{B}_i \\ \tilde{C}_i & \tilde{D}_i \end{pmatrix}. \quad (\text{A.2})$$

Next we solve for the meromorphic periods in terms of the holomorphic ones, as

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★ The one possible exception to this polynomial character of  $\tilde{U}_i$  may (but need not) occur for the  $SU(N_c)$  gauge groups, under the circumstances explained in [21].



in equation (5.3),

$$\begin{pmatrix} \Omega_{r+1} \\ \vdots \\ \Omega_{2r} \end{pmatrix} = (\det M) \tilde{B}_i^{-1} \cdot \frac{\partial}{\partial u_i} \begin{pmatrix} \Omega_1 \\ \vdots \\ \Omega_r \end{pmatrix} - \tilde{B}_i^{-1} \tilde{A}_i \begin{pmatrix} \Omega_1 \\ \vdots \\ \Omega_r \end{pmatrix}, \quad (\text{A.3})$$

and substitute into the remaining equations, as done in (5.4). After some rearrangements we find<sup>†</sup>

$$\begin{aligned} & \frac{\partial^2}{\partial u_i^2} \begin{pmatrix} \Omega_1 \\ \vdots \\ \Omega_r \end{pmatrix} - \left[ \frac{\partial \tilde{B}_i}{\partial u_i} \tilde{B}_i^{-1} + (\det M)^{-1} \tilde{A}_i \right. \\ & \quad \left. + (\det M)^{-1} \tilde{B}_i \tilde{D}_i \tilde{B}_i^{-1} - (\det M)^{-1} \frac{\partial}{\partial u_i} \det M \right] \frac{\partial}{\partial u_i} \begin{pmatrix} \Omega_1 \\ \vdots \\ \Omega_r \end{pmatrix} \\ & \quad + \left[ (\det M)^{-2} \tilde{B}_i \tilde{D}_i \tilde{B}_i^{-1} \tilde{A}_i - (\det M)^{-2} \tilde{B}_i \tilde{C}_i \right. \\ & \quad \left. + (\det M)^{-1} \frac{\partial \tilde{B}_i}{\partial u_i} \tilde{B}_i^{-1} \tilde{A}_i - (\det M)^{-1} \frac{\partial \tilde{A}_i}{\partial u_i} \right] \begin{pmatrix} \Omega_1 \\ \vdots \\ \Omega_r \end{pmatrix} = 0. \end{aligned} \quad (\text{A.4})$$

Obviously, equation (A.4) reduces to (5.5) upon formally setting  $\det M = 1$  and eliminating the tildes, as this corresponds to leaving  $\det M$  in the denominator in the right-hand side of equation (A.1).

Below we list a *Mathematica* programme that computes the matrices  $\tilde{A}_i$ ,  $\tilde{B}_i$ ,  $\tilde{C}_i$  and  $\tilde{D}_i$ , as well as  $\det M$ , for the  $SO(N_c)$  gauge theory with  $N_f$  massive multiplets, when  $N_f < N_c - r - 2$ . It can be easily modified to apply to the  $SU(N_c)$  and  $Sp(N_c)$  gauge groups. For typographical reasons, the names of some variables in the body of the programme have been changed with respect to the paper. Thus,  $P_n$  stands for  $\Omega_n^{(+1/2)}$ ,  $Q_n$  for  $\Omega_n^{(-1/2)}$ , “mu” for  $\mu$ , “nf” for  $N_f$ , “nc” for  $N_c$ ,  $R$  for  $\Delta(u_i, m_j, \Lambda)$ ,  $TU[i]$  for  $\tilde{U}_i$ ,  $TA[i]$  for  $\tilde{A}_i$ ,  $TB[i]$  for  $\tilde{B}_i$ ,  $TC[i]$  for  $\tilde{C}_i$ , and  $TD[i]$  for

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<sup>†</sup> No summation over  $i$  is implied here.

$\tilde{D}_i$ . The names of other variables used are explained in the programme. All the required equations are taken from the body of the paper.

```
(* Input the data

nf=number of flavours

nc=number of colours

d= 4 if nc is even, d= 2 if nc is odd

r=rank=nc/2 if nc is even, r=rank=(nc-1)/2 if nc is odd

mu= -1/2

m[1], m[2], ..., m[nf]: bare masses of the multiplets

T[0], T[1], ..., T[nf]: symmetric polynomials in the squared
masses

u[0], u[2], ..., u[2r-2]: moduli

and run *)

(* Known moduli for SO(nc) are the highest one, which is 1, and
all the odd ones, which vanish *)

Do[u[j]=0, {j, 1, 2r-1, 2}]

u[2r]=1;

(* Definition of the curve W. For simplicity the quantum scale
has been set to 1 *)

p[x]=Sum[u[j] x ^ j, {j, 0, 2r}];

W[x]=Collect[Expand[p[x]^2-x^d Product[(x^2-m[j]^2), {j, 1, nf,
1}]], x];

Print["p[x] = ", p[x]];

Print["W[x] = ", W[x]];

(* Discriminant of the curve *)
```

```

DxW=Simplify[D[W[x], x]];

R=Factor[Simplify[Resultant[W[x], DxW, x]]];

Print["R = ", R]

(* Definition of the list of initial values IV[j] that will
close the recursions *)

Do[P[j]=IV[j], {j, 0, 4r-2, 2}]

initials=Table[IV[j], {j, 0, 4r-2, 2}];

Print["Initial values = ", initials]

(* Recursion relation for the P[j] *)

P[n_Integer]=1/(n+1-4r(2+mu)) (Sum[(-1)^(nf-i)
(n-4r+1-(1+mu)(2i+d)) T[nf-i] P[n+d-4r+2i], {i, 0, nf, 1}]+
2Sum[((1+mu)(j+2r)-(n-4r+1)) u[j] P[n-2r+j], {j, 0, 2r-2, 1}]+
Sum[((1+mu)(i+j)-(n-4r+1)) u[i] u[j] P[n+i+j-4r], {i, 0, 2r-2, 1},
{j, 0, 2r-2, 1}]);

(* The required values for the P[j] are computed first for the
sake of efficiency, then expressed in terms of the initial values,
and finally stored as the new variables PP[j] *)

Do[PP[j]=Collect[Simplify[P[j]], initials], {j, 0, 8r-4, 2}]

(* Recursion relation for the Q[j] in terms of the PP[j] *)

Q[n_Integer]=-1/(n+1-4r(1+mu)) (Sum[(i+j-4r) u[i] u[j] PP[n+i+j],
{i, 0, 2r-2, 1}, {j, 0, 2r-2, 1}]+ Sum[(-1)^(nf-i) (4r-2i-d)
T[nf-i] PP[n+d+2i], {i, 0, nf, 1}]+ 2Sum[(j-2r) u[j] PP[n+j+2r], {j,
0, 2r-2, 1}]);

(* Values of P[j] or Q[j] can always be expressed as linear
combinations of the above initial values, with coefficients that are
polynomials in the moduli *)

```

```

(* Computation of the M matrix relating the Q[j] to the P[j] *)
Do[QQ[j]=Collect[Simplify[Q[j]], initials], {j, 0, 4r-2, 2}]

M=Table[Coefficient[QQ[i], IV[j]], {i, 0, 4r-2, 2}, {j, 0, 4r-2,
2}];

(* Next we check that the determinant of the M matrix, DM,
contains the same factors as the discriminant R of the curve,
possibly with different multiplicities *)

DM=Factor[Det[M]];

Print["DM = ", DM]

(* Definition of modular derivatives of the Q[j] in terms of the
P[j] *)

DQ[n_Integer, j_Integer]=2Sum[u[i] PP[n+j+i], {i, 0, 2r, 1}];

(* Computation of the matrices DD[j] expressing modular
derivatives of the Q[j] as linear combinations of the P[j] *)

Do[
Do[ DQQ[n, j]=Collect[Simplify[DQ[n, j]], initials], {n, 0,
4r-2, 2}];

DD[j]=Table[Coefficient[DQQ[n, j], IV[i]], {n, 0, 4r-2, 2}, {i,
0, 4r-2, 2}],

{j, 0, 2r-2, 2}]

(* Inversion of M. We pull its determinant, DM, to the left-hand
side of the equations. SIM is the matrix of cofactors of M *)

SIM=Simplify[DM*Inverse[M]];

(* Computation of the matrices TU[j] of coefficients in the
expansion of the modular derivatives of the Q[j] as linear
combinations of the Q[j] *)

```

```

Do[TU[j]=Simplify[DD[j].SIM], {j, 0, 2r-2, 2}]

(* Partition of the TU[j] into submatrices: TA[j], TB[j],
TC[j], TD[j] *)

Do[
  TA[j]=TU[j][[Range[1, r], Range[1, r]]];
  TB[j]=TU[j][[Range[1, r], Range[r+1, 2r]]];
  TC[j]=TU[j][[Range[r+1, 2r], Range[1, r]]];
  TD[j]=TU[j][[Range[r+1, 2r], Range[r+1, 2r]]],
  {j, 0, 2r-2, 2}]

Do[
  Print["----- Below is modulus ", u[j], " -----"];
  Print["TA[, j,] = ", TA[j]];
  Print[""];
  Print["TB[, j,] = ", TB[j]];
  Print[""];
  Print["Det[TB[, j, ]] = ", Factor[Det[TB[j]]]];
  Print[""];
  Print["TC[, j,] = ", TC[j]];
  Print[""];
  Print["TD[, j,] = ", TD[j]];
  Print[""],
  {j, 0, 2r-2, 2}]

(* One can check that the determinants of the TB[j] contain the
factors present in the discriminant R of the curve. *)

```